

Finite presentations of centrally extended mapping class groups

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Abstract

We describe a finite presentation of $\mathcal{T}_{g,r}$ for $g \geq 3$. Here $\mathcal{T}_{g,r}$ is the universal central extension of the mapping class group of the surface of genus g with r -boundaries. We also investigate the case $g = 2$, and give an application.

Keywords mapping class group, central extension

1 Introduction and main results

The mapping class group $\mathcal{M}_{g,r}$ of the surface $\Sigma_{g,r}$ of genus g with r -boundaries has been heavily studied in the topic of many study on low-dimensional topology. In particular, in quantum topology, Witten [Wi] has made a prophetic discovery that the Chern-Simons quantum field theory of level k (with Wilson loops) produces 3-manifold invariants and (quantum) representations V_k of $\mathcal{M}_{g,r}$ with an “anomaly”; this prophecy has since been mathematically formulated for some cases (see, e.g., [BHMV, GM, Koh]) and the anomaly has been interpreted as a 2-framing [Ati] or a p_1 -structure [BHMV]. Because of the obstacles posed by p_1 -structures, the space V_k is not always some right module of $\mathcal{M}_{g,r}$, but one of a central extension of $\mathcal{M}_{g,r}$.

Such central extensions are thought to be rather complicated. In contract, the subadjacent groups $\mathcal{M}_{g,r}$ has been widely analysed with finite presentations (see, e.g., [FM, Ge2]). To see this, since $\mathcal{M}_{g,r}$ with $g \geq 3$ is perfect (see [Kor]), let us set up the universal central extension,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{T}_{g,r} \xrightarrow{\text{proj.}} \mathcal{M}_{g,r} \longrightarrow 0 \quad (\text{exact}), \quad (1)$$

associated with the group cohomology $H^2(\mathcal{M}_{g,r}; \mathbb{Z}) \cong \mathbb{Z}$, which can be computed in a combinatorial way cite[§6]Kor. A 2-cocycle τ_g corresponding to a generator of the center \mathbb{Z} has some difficulties, as mentioned in [Ati]; the results of Meyer [Mey] and Turaev [Tur] indicates that the quadruple $4\tau_g$ can be geometrically described as a signature from the modular group $Sp(2g; \mathbb{Z})$; however, known formulations of τ_g contain something algebraic. Actually, Turaev formula of τ_g [Tur] is a signature 2-cocycle with a modification using a Maslov index. For this reason, as shown in [GM], the structure of $\mathcal{T}_{g,r}$ can be described in terms of Lagrangian cobordisms.

1.1 Results; The cases of $g \geq 3$ and $r = 0, 1$.

This paper explicitly describes a finite presentation of $\mathcal{T}_{g,r}$ for $g \geq 3$, $r \geq 0$. The presentations with $r = 0, 1$ are as follows:

Theorem 1. (I) Let $g \geq 3$. The universal central extension $\mathcal{T}_{g,1}$ of $\mathcal{M}_{g,1}$ has a presentation such that the generators are $c_0, c_1, \dots, c_{2g+1}, u$ and the relations are as follows:

$$(\text{Braid relation}) \quad c_i c_j = c_j c_i, \quad \text{if } I(\gamma_i, \gamma_j) = 0, \quad (2)$$

$$c_i c_j c_i = c_j c_i c_j, \quad \text{if } I(\gamma_i, \gamma_j) = 1, \quad (3)$$

$$(3\text{-chain relation}). \quad (c_1 c_2 c_3)^4 (c_0 b_0)^{-1} = \mu, \quad c_i \mu = \mu c_i, \quad (4)$$

$$(\text{Lantern relation}). \quad c_0 b_2 b_1 (c_1 c_3 c_5 b_3)^{-1} = 1, \quad (5)$$

whereby I means the geometric intersection number in Figure 1, and the notation b_0, b_1, b_2, b_3 is in common to the Wajnaryb's presentation (see [FM, Theorem 5.3]): Precisely, we have

$$\begin{aligned} b_0 &= (c_4 c_3 c_2 c_1 c_1 c_2 c_3 c_4)^{-1} c_0 (c_4 c_3 c_2 c_1 c_1 c_2 c_3 c_4), \\ b_1 &= (c_4 c_5 c_3 c_4)^{-1} c_0 (c_4 c_5 c_3 c_4), \quad b_2 = (c_2 c_3 c_1 c_2)^{-1} b_1 (c_2 c_3 c_1 c_2), \\ b_3 &= (c_6 c_5 c_4 c_3 c_2 c_5^{-1} c_6^{-1} b_1 c_6 c_5 c_1^{-1} c_2^{-1} c_3^{-1} c_4^{-1})^{-1} c_0 (c_6 c_5 c_4 c_3 c_2 c_5^{-1} c_6^{-1} b_1 c_6 c_5 c_1^{-1} c_2^{-1} c_3^{-1} c_4^{-1}). \end{aligned}$$

(II) Furthermore, concerning the closed surface of genus $g \geq 3$, the group $\mathcal{T}_{g,0}$ can be presented as above with adding the following commutator relation:

$$(c_{2g} c_{2g-1} \cdots c_1 c_1 c_2 \cdots c_{2g}) c_{2g+1} = c_{2g+1} (c_{2g} c_{2g-1} \cdots c_1 c_1 c_2 \cdots c_{2g}). \quad (6)$$

This presentation is a lift of the Wajnaryb presentation of $\mathcal{M}_{g,r}$. To be precise, his presentation can be made exactly as the quotient of the above presentation by adding the relation $\mu = 1$; see [FM, Theorem 5.3]. Consequently, the symbols c_i and b_i in $\mathcal{M}_{g,r}$ can be interpreted as Dehn twists along the respective curves γ_i and β_i in Figure 1.

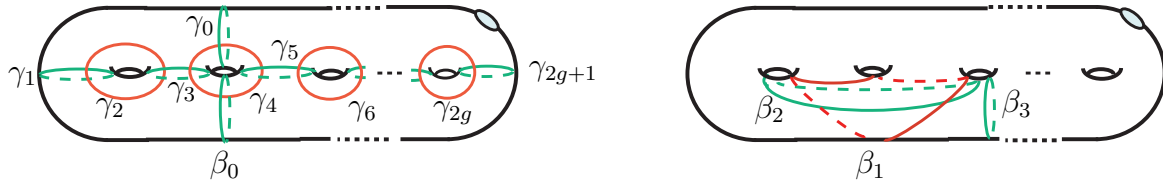


Figure 1: Generators of $\mathcal{M}_{g,*}$ with $g \geq 2$, $r \leq 1$, and the curves in the lantern relation with $g \geq 3$.

1.2 Result II; Punctured cases with $g \geq 3$, and the genus two case.

Furthermore, we can express a finite presentation of $\mathcal{T}_{g,r}$ with $g \geq 3$, $r \geq 1$. Following [Ge2], a triple $(i, j, k) \in \{1, \dots, 2g + r - 2\}^3$ is said to be *good*, if it satisfies $i \leq j \leq k$, or $j \leq k \leq i$ or $k \leq i \leq j$ without $i = j = k$. Here, consider the closed curves in $\Sigma_{g,r}$ depicted in Figure 2, and use the indices.

Theorem 2 (cf. [Ge2, Theorem 1]). *Let $g \geq 3$ and $r \geq 1$. The universal central extension $\mathcal{T}_{g,r}$ admits a presentation with generators $b, b_1, \dots, b_{g-1}, a_1, \dots, a_{2g+r-2}, \{c_{i,j}\}_{1 \leq i,j \leq 2g+r-2, i \neq j}$ and μ . Here the relations are as follows:*

- (i) “Handles”: $c_{2i,2i \mp 1} = c_{2i \pm 1,2i}$ for all i with $1 \leq i \leq g - 1$.
- (ii) “Braids”: $xy(xy^{-1})^{I(x,y)} = yx$ holds for all x, y among the generators with $I(x, y) \leq 1$, where $I(x, y)$ is the geometric intersection number of x, y according to Figure 2.
- (iii) “Stars”: $c_{i,j} c_{j,k} c_{k,i} = (a_i a_j a_k b)^3 \mu^{-1}$ for all good triples (i, j, k) . Here, we set $c_{\ell,\ell} = 1$.

(iv) “Centralization”: $[b, \mu] = [b_i, \mu] = [a_i, \mu] = [c_{i,j}, \mu] = 1$.

In analogy to Theorem 1, this presentation is a lift of Gervais’s one [Ge2] of $\mathcal{M}_{g,r}$. To be precise, [Ge2, Theorem 1] with $g \geq 1$, $r \geq 1$ says that the group with the presentation subject to $\mu = 1$ is isomorphic to $\mathcal{M}_{g,r}$.

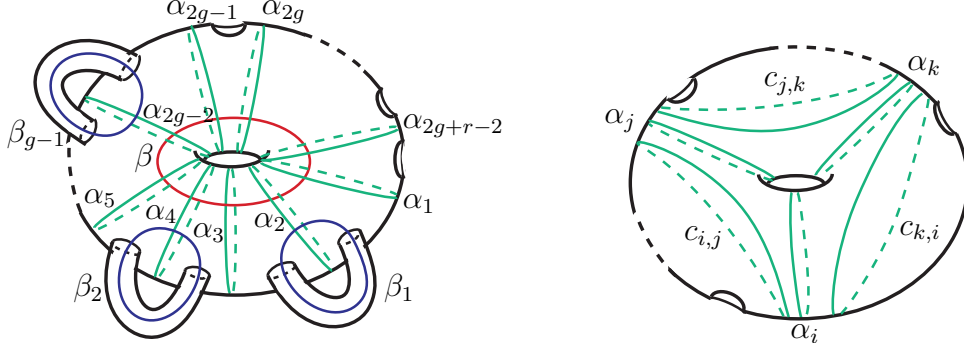


Figure 2: Gervais’s generators of $\mathcal{M}_{g,r}$ with $g \geq 2$, $r \geq 1$.

Finally, we focus on the case $g = 2$. Although \mathcal{T}_2 is not perfect, we will give a presentation of a \mathbb{Z} -central extension \mathcal{T}_2 of \mathcal{M}_2 (see the proof of Theorem 4 for the definition).

Theorem 3. *The \mathbb{Z} -central extension \mathcal{T}_2 of \mathcal{M}_2 has a presentation with generators c_1, c_2, c_3, c_4 and c_5 . Here, the relations are defined by the two previous ones (2), (3) and the following:*

$$((c_1 c_2 c_3)^4 c_5^{-2})^2 = (c_5 c_4 c_3 c_2 c_1 c_1 c_2 c_3 c_4 c_5)^2, \quad (7)$$

$$[c_5 c_4 c_3 c_2 c_1 c_1 c_2 c_3 c_4 c_5, c_1] = 1. \quad (8)$$

1.3 Concluding remarks

To summarise the results, we emphasise that most of the central extensions $\mathcal{T}_{g,r}$ are dealt with concretely as finitely presented groups, since the remaining case of genus one is widely known (see [FM, §6.3]). Moreover, the results are of use for checking whether a linear representation $\rho : \mathcal{T}_{g,r} \rightarrow GL_n(\mathbb{C})$ is well-defined or not: For instance, if $r = 0$ or $r = 1$, according to Theorem 1, it is important to check that the lantern relation and the 3-chain relation are sent to $c \cdot \text{id}_{\mathbb{C}^n}$ for some $c \in \mathbb{C}$. Furthermore, the theorems are also useful in normalizing (constant factors of) Kohn’s description [Koh] on 3-manifold invariants from $\rho_{\mathcal{G},k}$ and in computing concretely Lefschetz fibration invariants [N2], which are constructed from $\rho_{\mathcal{G},k}$.

2 Proof of the theorems and the associated group.

This section is dedicated to proving the theorems stated above. In this section, we assume that the reader has basic knowledge of group cohomology and the mapping class group explained in [FM, §1-5], [Kor].

The outline of the proofs is as follows. First, we review the group “ $\text{As}(\mathcal{D}_g^{\text{ns}})$ ” that is introduced in the study of the TQFT [MR] and show $\text{As}(\mathcal{D}_g^{\text{ns}}) \cong \mathbb{Z} \times \mathcal{T}_{g,0}$ for $g \geq 3$; see Theorem 4. As a corollary, we show that the group presented in Theorem 1 (II) is isomorphic to $\text{As}(\mathcal{D}_g^{\text{ns}})$ without the \mathbb{Z} -factor. Concerning Theorem 3, a similar discussion is applicable to

the genus two case. After that, Theorem 1 (I) is shown to follow from the universality of the central extensions and Harer-Ivanov stability on the second cohomologies between \mathcal{M}_g and $\mathcal{M}_{g,1}$. Next, Theorem 2 is proven by induction on $r \geq 1$. Actually, the group presented in Theorem 2 for $r = 1$ is shown to be isomorphic to that in Theorem 1, and the proof for $r \geq 2$ is similarly completed using a result of Harer-Ivanov stability. Incidentally, in an application of the proof, we propose (Proposition 6) a criterion for which the quantum representation is useful for some Lefschetz fibration invariants; see [N2] for details.

2.1 Preliminaries: the associated group

Here, we describe terminologies and state a key proposition (Theorem 4). First, denoting $\mathcal{M}_{g,0}$ by \mathcal{M}_g for short, we set the following three subsets:

$$\mathcal{D}_g := \{ \tau_\alpha \in \mathcal{M}_g \mid \alpha \text{ is an (unoriented) simple closed curve } \gamma \text{ in } \Sigma_g \}, \quad (9)$$

$$\mathcal{D}_g^{\text{ns}} := \{ \tau_\alpha \in \mathcal{D}_g \mid \alpha \text{ is a non-separating simple closed curve } \gamma \text{ in } \Sigma_g \}, \quad (10)$$

$$\mathcal{D}_g^{(k)} := \{ \tau_\alpha \in \mathcal{D}_g \mid \text{The complement } \Sigma_g \setminus \alpha \text{ is homeomorphic to } \Sigma_{k,1} \sqcup \Sigma_{g-k,1} \},$$

where the symbol τ_α is the (positive) Dehn twist along α . Next, for $Z = \mathcal{D}_g$ or $Z = \mathcal{D}_g^{\text{ns}}$, we will analyze the group $\text{As}(Z)$, which is considered in [MR]. Here $\text{As}(Z)$ is defined to be the abstract group generated by symbols e_z with $z \in Z$ subject to the relation $e_{w^{-1}zw} = e_w^{-1}e_z e_w$ and is called *the associated group*. Note that the inclusion $\mathcal{D}_g \hookrightarrow \mathcal{M}_g$ gives rise to a group epimorphism $\mathcal{E} : \text{As}(\mathcal{D}_g) \rightarrow \mathcal{M}_g$ by definition, and we further can describe the equality as

$$g e_z g^{-1} = e_{\mathcal{E}(g e_z g^{-1})} \in \text{As}(\mathcal{D}_g), \quad \text{for any } z \in \mathcal{D}_g, \quad g \in \text{As}(\mathcal{D}_g), \quad (11)$$

which is easily verified by induction on the word length of g . The reader should keep in mind this equality, since we will use in many times. For example, as a result of (11), the kernel $\text{Ker}(\mathcal{E})$ is contained in the center. In summary, we have

$$0 \longrightarrow \text{Ker}(\mathcal{E}) \longrightarrow \text{As}(\mathcal{D}_g) \xrightarrow{\mathcal{E}} \mathcal{M}_g \longrightarrow 0, \quad (\text{central extension}). \quad (12)$$

In addition, we can explicitly determine the central extension $\text{As}(Z)$ as follows:

Theorem 4. (cf. [Ge1, Theorem C]) (I) If $g \geq 3$, there are isomorphisms $\text{As}(\mathcal{D}_g) \cong \mathcal{T}_g \times \mathbb{Z}^{[g/2]+2}$ and $\text{As}(\mathcal{D}_g^{\text{ns}}) \cong \mathcal{T}_g \times \mathbb{Z}$.

(II) If $g = 2$, there is a central \mathbb{Z} -extension $\mathcal{T}_2 \rightarrow \mathcal{M}_2$ for which $\text{As}(\mathcal{D}_g) \cong \mathcal{T}_2 \times \mathbb{Z}^2$ and $\text{As}(\mathcal{D}_g^{\text{ns}}) \cong \mathcal{T}_2$ hold.

Here, we shall refer to [Ge1, Theorem C] which was claimed to be the same statement on $\text{As}(\mathcal{D}_g^{\text{ns}})$ and discussed an infinite presentation of $\mathcal{T}_{g,r}$ from the p^1 -structure. But the proofs might contain some gaps (e.g., functoriality with respect to lantern relations of $\widetilde{\mathcal{M}}_g$ [Ge1, Theorems C, 4.1]. Furthermore, the case $g = 3$). In contrast, the proofs in this paper do not use combinatorial computation (cf. [Ge1]) or the Maslov index as in [Ati, GM, Tur], but only basic knowledge of group cohomology of degree 2.

Let us begin by describing the concepts of lantern relations in $\text{As}(\mathcal{D}_g)$. If $g \geq 3$, consider two elements of the form

$$\kappa_{3\text{-chain}} := (e_{c_1} e_{c_2} e_{c_3})^4 e_{c_0}^{-1} e_{b_0}^{-1}, \quad \kappa_{\text{lantern}} := e_{c_1}^{-1} e_{c_3}^{-1} c_{c_5}^{-1} e_{b_3}^{-1} e_{b_0} e_{b_1} e_{b_2} \in \text{As}(\mathcal{D}_g),$$

where b_i and c_i are the Dehn twists along the respective curves β_i and γ_i in Figure 1; otherwise, if $g = 2$, define $\kappa_{3\text{-chain}}$ to be $(e_{c_1} e_{c_2} e_{c_3})^4 e_{c_5}^{-2}$ and κ_{lantern} to be $1_{\text{As}(\mathcal{D}_g)}$. As is wellknown, $\mathcal{E}(\kappa_{3\text{-chain}})$ and $\mathcal{E}(\kappa_{\text{lantern}})$ are respectively the identity in \mathcal{M}_g , which is commonly called the *3-chain relation* and the *lantern relation*. Furthermore, for $k < g/2$, define seven curves $\alpha_k, \beta_k, \gamma_k, \delta_k, x_k, y_k, z_k$ in Σ_g illustrated in Figure 3, and put the product $\mathcal{L}_k := e_{\alpha_k} e_{\beta_k} e_{\gamma_k} e_{\delta_k} e_{x_k}^{-1} e_{y_k}^{-1} e_{z_k}^{-1} \in \text{As}(\mathcal{D}_g)$. The lantern relations in \mathcal{M}_g tell us that these κ_{lantern} and \mathcal{L}_k lie in $\text{Ker}(\mathcal{E})$.

In addition, for $\dagger = 0, 1, \dots, [g/2]$ or $\dagger = \text{ns}$, we define a homomorphism $\epsilon_{\dagger} : \text{As}(\mathcal{D}_g) \rightarrow \mathbb{Z}$ by setting $\epsilon_{\dagger}(e_x) = 1 \in \mathbb{Z}$ if $x \in \mathcal{D}_g^{\dagger}$ and by setting $\epsilon_{\dagger}(e_x) = 0 \in \mathbb{Z}$ otherwise. Since the orbit decomposition of the conjugate action $\mathcal{D}_g \curvearrowright \mathcal{M}_g$ is presented as $\mathcal{D}_g = \mathcal{D}_g^{\text{ns}} \cup (\bigsqcup_{0 \leq k \leq g/2} \mathcal{D}_g^{(k)})$, it follows from (11) that the sum $(\bigoplus \epsilon_j) \oplus \epsilon_{\text{ns}}$ gives an abelianization $H_1(\text{As}(\mathcal{D}_g)) \cong \mathbb{Z}^{[\frac{g}{2}] + 2}$.

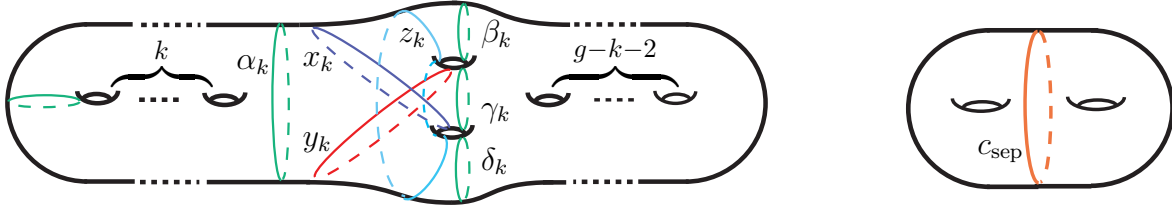


Figure 3: The k -th lantern relation \mathcal{L}_k in \mathcal{M}_g with $g > 2$, $k \geq 0$, and the curve c_{sep} with $g = 2$.

Furthermore, let us review [EN, Theorem 4.2]. Notice from (11) that, if a tuple $\mathbf{z} = (z_1, \dots, z_m) \in (\mathcal{D}_g)^m$ satisfies $z_1 \cdots z_m = 1_{\mathcal{M}_g}$, the product $e_{z_1} \cdots e_{z_m} \in \text{As}(\mathcal{D}_g)$ lies in the central kernel $\text{Ker}(\mathcal{E} : \text{As}(\mathcal{D}_g) \rightarrow \mathcal{M}_g)$. Furthermore, we can construct a closed oriented 4-manifold $E_{\mathbf{z}}$ from such a tuple $\mathbf{z} \in (\mathcal{D}_g)^m$, which is called a Lefschetz fibration (see, e.g., [EN] for the definition). Inspired by [Ge1] and the Hopf theorem on H_2 , Endo and Nagami [EN, Definition 3.3 and Proposition 3.6] constructed a homomorphism $I_g : \text{Ker}(\mathcal{E}) \rightarrow \mathbb{Z}$ that enjoys the following property:

Theorem 5 ([EN, Theorem 4.2 and Propositions 3.10-3.12]). *I_g satisfies*

$$I_g(e_{z_1} \cdots e_{z_m}) = \sigma_{\mathbf{z}} + m - m_{\text{ns}} \in \mathbb{Z},$$

for any m -tuple $(z_1, \dots, z_m) \in (\mathcal{D}_g)^m$ with $z_1 \cdots z_m = 1_{\mathcal{M}_g} \in \mathcal{M}_g$. Here $m_{\text{ns}} \in \mathbb{Z}$ is the number of j 's with $z_j \in \mathcal{D}_g^{\text{ns}}$, and $\sigma_{\mathbf{z}}$ is the signature of the associated 4-manifold $E_{\mathbf{z}}$.

Moreover, $I_g(\kappa_{3\text{-chain}}) = -6$; Further, if $g \geq 3$, then $I_g(\kappa_{\text{lantern}}) = 1$.

2.2 Proofs of theorems for $g \geq 3$.

Proof of Theorem 4 (I). For the proof of (I), we briefly analyze the group $\text{As}(\mathcal{D}_g)$ with $g \geq 3$ as a special tool in quandle theory (see [N1, Proposition A.4.]). Consider a homomorphism $\mathfrak{s}_j : \mathbb{Z} \rightarrow \text{As}(\mathcal{D}_g)$ which sends n to $(\mathcal{L}_j)^n$. This \mathfrak{s}_j is a section of ϵ_j , and the image is contained in the center $\text{Ker}(\mathcal{E})$ because of (11). Therefore, the semiproduct structure associated with \mathfrak{s}_j

is trivial; hence we have the decomposition $\text{As}(\mathcal{D}_g) \cong \tilde{\mathcal{M}}_g \times \mathbb{Z}^{[\frac{g}{2}]+2}$ for some central extension $\tilde{\mathcal{M}}_g$ of \mathcal{M}_g . Considering Kunneth formula on H_1 , this $\tilde{\mathcal{M}}_g$ is perfect. Furthermore, from the construction of the decomposition, we similarly have $\text{As}(\mathcal{D}_g^{\text{ns}}) \cong \tilde{\mathcal{M}}_g \times \mathbb{Z}$.

Hence, it is sufficient to show $\tilde{\mathcal{M}}_g \cong \mathcal{T}_g$. To this end, notice from the infraction-restriction exact sequence (cf. (17) below) that the kernel of $\tilde{\mathcal{M}}_g \rightarrow \mathcal{M}_g$ is a quotient of $H_2(\mathcal{M}_g)$, since $\tilde{\mathcal{M}}_g$ is perfect. Here, we claim that the kernel contains \mathbb{Z} . Actually, it follows from Theorem 5 that the sum of the homomorphisms $(I_g \oplus \epsilon_{\text{ns}}) \oplus (\bigoplus \epsilon_j) : \text{Ker}(\mathcal{E}) \rightarrow \mathbb{Z}^{[\frac{g}{2}]+3}$ is of order 4 at most (cf. the quadruple $4\tau_g$ mentioned in §1).

We will show that $\tilde{\mathcal{M}}_g \cong \mathcal{T}_g$ is a result of the claim. First, if $g \geq 4$, the kernel of $\tilde{\mathcal{M}}_g \rightarrow \mathcal{M}_g$ must be $H_2(\mathcal{M}_g) \cong \mathbb{Z}$; hence, the universality of the central extensions implies $\tilde{\mathcal{M}}_g \cong \mathcal{T}_g$ as desired. Next, if $g = 3$, the kernel is a quotient of $H_2(\mathcal{M}_3) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ [Sak, Corollary 4.10]; hence, $\tilde{\mathcal{M}}_g$ surjects onto \mathcal{T}_g and the kernel is either 0 or $\mathbb{Z}/2$. Thus, $\text{Ker}(\mathcal{E}) \cap \text{As}(\mathcal{D}_g^{\text{ns}})$ is either \mathbb{Z}^2 or $\mathbb{Z}^2 \oplus \mathbb{Z}/2$. However, recalling the Wajnaryb presentation, \mathcal{M}_g is the quotient of $\text{As}(\mathcal{D}_g^{\text{ns}})$ subject to only the two central elements $\kappa_{3\text{-chain}} = \kappa_{\text{lantern}} = 1$; Hence, $\text{Ker}(\mathcal{E}) \cap \text{As}(\mathcal{D}_g^{\text{ns}})$ is nothing but \mathbb{Z}^2 , which implies $\tilde{\mathcal{M}}_g \cong \mathcal{T}_g$ as required. \square

We are now in a position to prove Theorem 1. Notice from Theorem 5 that

$$I_g(\kappa_{3\text{-chain}}\kappa_{\text{lantern}}^{10}) = 4, \quad \epsilon_{\text{ns}}(\kappa_{3\text{-chain}}\kappa_{\text{lantern}}^{10}) = 0. \quad (13)$$

Since the cokernel of $(I_g \oplus \epsilon_{\text{ns}}) \oplus (\bigoplus \epsilon_j)$ is $\mathbb{Z}/4$ as in the previous proof, this $\kappa_{3\text{-chain}}\kappa_{\text{lantern}}^{10}$ is a generator of the center $\mathbb{Z} = \text{Ker}(\mathcal{T}_g \rightarrow \mathcal{M}_g)$ in (1). Furthermore, let us define the epimorphism

$$\theta_r : \mathcal{M}_{g,r} \longrightarrow \mathcal{M}_{g,r-1} \quad (14)$$

induced by gluing a disc to the boundary component of $\Sigma_{g,1}$.

Proof of Theorem 1 with $g \geq 3$. Let $r = 0$ or 1 , let $\mathcal{G}_{g,r}$ be the group with the presentation given in Theorem 1, and let $q_r : \mathcal{G}_{g,r} \rightarrow \mathcal{M}_{g,r}$ be the quotient map by adding the relation $\mu = 1$. Noting the relation (4), the map q_r is a central extension with fiber \mathbb{Z} .

(II) Using (11), we can verify that the correspondence $c_i \mapsto e_{\gamma_i}\kappa_{\text{lantern}}$, $\mu \mapsto \kappa_{3\text{-chain}}\kappa_{\text{lantern}}^{10}$ defines a homomorphism $\psi : \mathcal{G}_{g,0} \rightarrow \text{As}(\mathcal{D}_g^{\text{ns}})$ over \mathcal{M}_g . The image is included in $\mathcal{T}_g = \text{Ker}(\epsilon_{\text{ns}} : \text{As}(\mathcal{D}_g^{\text{ns}}) \rightarrow \mathbb{Z})$ by definition. Hence, since \mathcal{T}_g is universal, ψ must be an isomorphism $\mathcal{G}_{g,0} \cong \mathcal{T}_g$ as required.

(I) From the definition of (14), we have a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{G}_{g,1} & \xrightarrow{q_1} & \mathcal{M}_{g,1} \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{proj.} & & \downarrow \theta_1 \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{G}_{g,0} & \xrightarrow{q_0} & \mathcal{M}_g \longrightarrow 0 \end{array} \quad \begin{array}{l} \text{(central extension)} \\ \\ \text{(central extension).} \end{array} \quad (15)$$

From the definitions of $\mathcal{G}_{r,*}$, the left vertical map is isomorphic. As it is a part of the Harer-Ivanov stability with $g \geq 3$ (see [Kor, §6]), the right induced map $\theta_1^* : H^2(\mathcal{M}_g; \mathbb{Z}) \rightarrow H^2(\mathcal{M}_{g,1}; \mathbb{Z})$ is an isomorphism on \mathbb{Z} . Since $\mathcal{G}_{g,0} \cong \mathcal{T}_g$, the universality of the central extensions implies the desired isomorphism $\mathcal{G}_{g,1} \cong \mathcal{T}_{g,1}$. \square

Now, let us turn into proving Theorem 2 for the punctured groups. Let us denote by $\mathcal{G}'_{g,r}$ the group with the presentation. Recalling the quotient $\mathcal{G}'_{g,r}/\langle \mu = 1 \rangle \cong \mathcal{M}_{g,r}$ as a result of [Ge2, Theorem 1], we see from (iv) that the projection $\mathcal{G}'_{g,r} \rightarrow \mathcal{M}_{g,r}$ is a central \mathbb{Z} -extension.

Proof of Theorem 2. As the first step of the induction on r , we let $g \geq 3$, $r = 1$ and will observe a diagram similar to (15). Consider the map $q_1 : \mathcal{G}'_{g,1} \rightarrow \mathcal{M}_{g,1}$ which takes each generators α of $\mathcal{G}'_{g,1}$ without μ to the corresponding Dehn twist τ_α . It immediately follows from [Ge2, Lemma 5] that the quotient of q_1 modulo $\mu = 1$ is isomorphic to $\mathcal{M}_{g,1}$, and that q_1 is a central \mathbb{Z} -extension from the definition of μ . Furthermore, using the homomorphism θ_1 in (15), consider the correspondence to $\text{As}(\mathcal{D}_g)$ defined by

$$\gamma \longmapsto e_{\theta_1(\gamma)} \kappa_{\text{lantern}}^{-\epsilon_{\text{ns}}(\theta_1(\gamma))} \left(\mathcal{L}_1^{\epsilon_1(\theta_1(\gamma))} \dots \mathcal{L}_{[g/2]}^{\epsilon_{[g/2]}(\theta_1(\gamma))} \right)^{-1}, \quad \mu \mapsto \kappa_{3\text{-chain}} \kappa_{\text{lantern}}^{10}.$$

Here, γ runs over the generators in Theorem 2, and \mathcal{L}_k is the central element defined in §2.3. Note [EN, Proposition 3.13] that the homomorphism $I_g : \text{Ker}(\mathcal{E}) \rightarrow \mathbb{Z}$ in Theorem 5 (resp. ϵ_{ns}) sends the star relation (iii) after projecting onto \mathcal{M}_g to $5 - N_{i,j,k}$ (resp. $9 - N_{i,j,k}$), where $N_{i,j,k}$ is the number of $\{c_{i,j}, c_{j,k}, c_{k,i} \mid c_{x,y} \text{ is separating.}\}$. Hence, compared with (13), the correspondence defines a map $\text{proj} : \mathcal{G}'_{g,1} \rightarrow \text{As}(\mathcal{D}_g)$ as a centrally extended homomorphism over θ_1 . Note that the image is $\text{Ker}(\oplus_{\dagger} : \text{As}(\mathcal{D}_g) \rightarrow \mathbb{Z}^{[g/2]+2}) \cong \mathcal{T}_g$ by definition. By performing a diagram chasing similar to (15) and the universality of central extensions, $\mathcal{G}'_{g,1}$ must be $\mathcal{T}_{g,1}$.

Finally, we now complete the proof with $r \geq 2$. Consider the canonical surjection $p_r : \mathcal{G}'_{g,r} \rightarrow \mathcal{G}'_{g,r-1}$ obtained from the presentations. Then, the quotient p_r modulo $\mu = 1$ is identified with $\theta_r : \mathcal{M}_{g,r} \rightarrow \mathcal{M}_{g,r-1}$ in (14). In addition, consider the injection $\iota_r : \mathcal{M}_{g,r-1} \rightarrow \mathcal{M}_{g,r}$ induced by gluing a two holed disc to a boundary component of $\Sigma_{g,r}$. From the Harer-Ivanov stability (see [Kor]), the induced map $\iota_r^* : H^2(\mathcal{M}_{g,r}; \mathbb{Z}) \rightarrow H^2(\mathcal{M}_{g,r-1}; \mathbb{Z})$ is known to be an isomorphism on \mathbb{Z} . Furthermore, noting that $\theta_r \circ \iota_r = \text{id}$, the induced map $\theta_r^* : H^2(\mathcal{M}_{g,r-1}; \mathbb{Z}) \rightarrow H^2(\mathcal{M}_{g,r}; \mathbb{Z})$ is also an isomorphism on \mathbb{Z} . Consequently, the surjection p_r induces an isomorphism on H^2 . Hence, since $H^2(\mathcal{G}'_{g,r-1}; \mathbb{Z}) \cong 0$ by induction on r , we have $H^2(\mathcal{G}'_{g,r}; \mathbb{Z}) \cong 0$. In conclusion, the central \mathbb{Z} -extension $\mathcal{G}'_{g,r}$ is also universal, that is, $\mathcal{G}'_{g,r} \cong \mathcal{T}_{g,r}$ as desired. \square

2.3 The case of two genus, and an application

This subsection is devoted to giving the proofs with $g = 2$.

Proof of Theorem 4 (II). First, we will show the isomorphism (16) below. Consider two homomorphisms $\mathfrak{s}, \mathfrak{t}$ defined by setting

$$\begin{aligned} \mathfrak{s} : \mathbb{Z}^2 &\longrightarrow \text{As}(\mathcal{D}_g); \quad (n, m) \longmapsto ((e_{c_1} e_{c_2})^6 e_{c_{\text{sep}}}^{-1})^n (\mathcal{L}_0)^m, \\ \mathfrak{t} : \mathbb{Z}^3 &\longrightarrow \mathbb{Z}^2; \quad (a, b, c) \longmapsto (12a + c, b). \end{aligned}$$

Here \mathcal{L}_0 and c_{sep} are Dehn twists along the curves described in Figure 3, respectively. Denote by \mathfrak{u} the composite $\mathfrak{t} \circ (\epsilon_{\text{ns}} \oplus \epsilon_0 \oplus \epsilon_1) : \text{As}(\mathcal{D}_g) \rightarrow \mathbb{Z}^2$. Then the 2-chain relation in \mathcal{M}_2 implies that this \mathfrak{s} is a splitting of \mathfrak{u} . Since the image of \mathfrak{u} is contained in the center of $\text{As}(\mathcal{D}_g)$, the semiproduct from \mathfrak{u} is trivial, i.e.,

$$\text{As}(\mathcal{D}_g) \cong \mathbb{Z}^2 \times \text{Ker}(\mathfrak{u}). \tag{16}$$

Moreover, the inclusion $\text{As}(\mathcal{D}_g^{\text{ns}}) \subset \text{As}(\mathcal{D}_g)$ leads to $\text{As}(\mathcal{D}_g^{\text{ns}}) \cong \text{Ker}(\mathfrak{u})$ in a similar way as in the case $g \geq 3$. Denote this central extension $\text{Ker}(\mathfrak{u})$ over \mathcal{M}_2 by \mathcal{T}_2 .

For the proof that the center of \mathcal{T}_2 is \mathbb{Z} , it suffices to show

$$\text{Ker}(\mathcal{E}) \cap \text{Ker}(\mathbf{u}) = \text{Ker}(\mathcal{E}) \cap \text{As}(\mathcal{D}_g^{\text{ns}}) \cong \mathbb{Z},$$

where \mathcal{E} is the central extension $\text{As}(\mathcal{D}_g) \rightarrow \mathcal{M}_g$ in (12). To show the isomorphism, recall $H_1(\text{As}(\mathcal{D}_g)) \cong \mathbb{Z}^3$ mentioned above, and the basic facts $H_1(\mathcal{M}_2) \cong \mathbb{Z}/10$ and $H_2(\mathcal{M}_2) \cong \mathbb{Z}/2$; see [FM, Kor]. Then the infraction-restriction exact sequence for \mathcal{E} can be written as

$$H_2(\text{As}(\mathcal{D}_2)) \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Ker}(\mathcal{E}) \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}/10 \longrightarrow 0, \quad (\text{exact}). \quad (17)$$

Therefore, $\text{Ker}(\mathcal{E})$ is either \mathbb{Z}^3 or $\mathbb{Z}^3 \oplus \mathbb{Z}/2$. Furthermore, from (16), $\text{Ker}(\mathcal{E}) \cap \text{As}(\mathcal{D}_g^{\text{ns}})$ is either \mathbb{Z} or $\mathbb{Z} \oplus \mathbb{Z}/2$. From the presentation of \mathcal{M}_2 , \mathcal{M}_2 is isomorphic to the quotient $\text{As}(\mathcal{D}_g^{\text{ns}})/\langle \kappa_{3\text{-chain}} \rangle$. Hence, $\text{Ker}(\mathcal{E}) \cap \text{As}(\mathcal{D}_g^{\text{ns}})$ turns out to be \mathbb{Z} as desired. \square

Proof of Theorem 3. Denote the group presented in the statement by \mathcal{G} . Then, the quotient of \mathcal{G} subject to $(c_1 c_2 c_3)^4 c_5^{-2} = 1$ is \mathcal{M}_2 . Further, from (11), the quotient map $\mathcal{G} \rightarrow \mathcal{M}_2$ is a central \mathbb{Z} -extension. A diagram chasing of (17) easily reveals that $(\kappa_{3\text{-chain}})^2$ is equal to $(e_{\gamma_5} e_{\gamma_4} e_{\gamma_3} e_{\gamma_2} e_{\gamma_1} e_{\gamma_1} e_{\gamma_2} e_{\gamma_3} e_{\gamma_4} e_{\gamma_5})^2$ contained in the center. As before, the correspondence $c_i \mapsto e_{\gamma_i}$ defines an epimorphism between central \mathbb{Z} -extensions $\mathcal{G} \rightarrow \text{As}(\mathcal{D}_g^{\text{ns}})$ over \mathcal{M}_g , which is an isomorphism. Hence, the above result $\text{As}(\mathcal{D}_g^{\text{ns}}) \cong \mathcal{T}_2$ immediately leads to the conclusion. \square

We conclude this paper by discussing right \mathcal{T}_g -modules and tuples $(z_1, \dots, z_m) \in (\mathcal{D}_g)^m$ with $z_1 \cdots z_m = 1_G$. Recall that the product $e_{z_1} \cdots e_{z_m} \in \text{As}(\mathcal{D}_g)$ lies in the center $\text{Ker}(\mathcal{E})$. Furthermore, in the study of Lefschetz fibration invariants, it is important to verify whether the identity $e_{z_1} \cdots e_{z_m} = \text{id}_M$ holds or not (see [N2, §3.2] for the details). To do so in an easy way, we will show that the identity can be established by the central elements $\kappa_{3\text{-chain}}$, κ_{lantern} and signature of 4-dimensional Lefschetz fibrations: Precisely,

Proposition 6. *Let regard a right \mathcal{T}_g -module M as an $\text{As}(\mathcal{D}_g)$ -module via the isomorphism $\text{As}(\mathcal{D}_g) \cong \mathcal{T}_g \times \mathbb{Z}^{[g/2]+2}$. Denote the associated map $\text{As}(\mathcal{D}_g) \rightarrow \text{End}(M)$ by ρ .*

(I) *If $g \geq 3$, then the identity*

$$\rho(e_{z_1} \cdots e_{z_m}) = \rho(\kappa_{3\text{-chain}})^{(\sigma_{\mathbf{z}}+m)/4} \rho(\kappa_{\text{lantern}})^{(5\sigma_{\mathbf{z}}+5m-2m_{\text{ns}})/2} \in \text{End}(M) \quad (18)$$

holds for any tuple $(z_1, \dots, z_m) \in (\mathcal{D}_g)^m$ with $z_1 \cdots z_m = 1$. Here, we fix an appropriate sign in the identity, and the notation $\sigma_{\mathbf{z}}$, $m_{\text{ns}} \in \mathbb{N}$ are the same as in Theorem 5.

In particular, if the right hand side is the identity, then $\rho(e_{z_1} \cdots e_{z_m}) = \text{id}_M$.

(II) *Similarly, if $g = 2$, then the identity $\rho(e_{z_1} \cdots e_{z_m}) = \rho(\kappa_{3\text{-chain}})^{(\sigma_{\mathbf{z}}+m-m_{\text{ns}})/6} \in \text{End}(M)$ holds for any tuple $\mathbf{z} = (z_1, \dots, z_m) \in (\mathcal{D}_g)^m$ satisfying $z_1 \cdots z_m = 1$.*

Proof. Since the element $e_{z_1} \cdots e_{z_m}$ is contained in $\text{Ker}(\mathcal{E})$, from Theorem 4, there exist $N_C, N_L \in \mathbb{Z}$ for which the following holds:

$$(e_{z_1} \cdots e_{z_m})^{-1} (\kappa_{3\text{-chain}})^{N_C} (\kappa_{\text{lantern}})^{N_L} \in \text{As}\left(\bigcup_{0 \leq k \leq g/2} \mathcal{D}_g^{(k)}\right) \quad (19)$$

First, we show (I). Considering the length $m_{\text{ns}} = \epsilon_{\text{ns}}(e_{z_1} \cdots e_{z_m}) \in \mathbb{Z}$, we obtain $m_{\text{ns}} = N_L + 10N_C$. Furthermore, recall from Theorem 5 that the homomorphism I_g satisfies $I_g(\kappa_{3\text{-chain}}) =$

-6 and $I_g(\kappa_{\text{lantern}}) = 1$; The former statement in Theorem 5 yields $\sigma_{\mathbf{z}} = N_L + 6N_C + m - m_{\text{ns}}$, leading to the solution $N_C = (5\sigma_{\mathbf{z}} + 5m - 2m_{\text{ns}})/2$ and $N_L = (\sigma_{\mathbf{z}} + m)/4$. Hence, applying ρ to (19) implies the desired equality (18) as claimed.

(II) Finally, we deal with the case $g = 2$. Notice that $\kappa_{\text{lantern}} = 1$ and $N_L = 0$ by definition. Similarly, the previous $I_g(\kappa_{3\text{-chain}}) = -6$ leads to the equality $\sigma_{\mathbf{z}} = 6N_C + m - m_{\text{ns}}$. Consequently, the solution $N_C = (\sigma_{\mathbf{z}} - m + m_{\text{ns}})/6$ follows from (19). Hence, the result $\text{As}(\mathcal{D}_2) \cong \mathcal{T}_2 \times \mathbb{Z}^2$ readily yields the required equality. \square

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